SPRING 2025 MATH 540: QUIZ 11

Name:

1. Show that if a natural number n is the sum of two rational squares it is also the sum of two integer squares. Hint: Use the theorem from class characterizing when in integer is the sum of two squares.

Solution. Suppose $n = (\frac{a}{b})^2 + (\frac{c}{d})^2$. We may assume the given fractions are reduced to lowest terms. Then $b^2d^2n = (ad)^2 + (cb)^2$. By the corollary to Fermat's two square theorem,

$$b^2d^2n = 2^{e_0}p_1^{e_1}\cdots p_r^{e_r}q_1^{f_1}\cdots q_s^{f_s},$$

for primes p_i, q_j, where each $p_i \equiv 1 \bmod 4$ and $q_j \equiv 3 \bmod 4$, and each $e_i, f_j \geq 0$, and each f_j even. Since all the primes in b^2d^2 appear to even powers, the FTA tells us the primes in n that are congruent to 3 mod 4 must appear with even exponents, so n is a sum of two squares.

2. Assume x, y, z is a Pythagorean triple. Show that at east one of x, y or z is divisible by 4.

Solution. Without loss of generality, we may assume the triple is primitive, and y is even. Then, we have $m, n > 0$ such that $x = m^2 - n^2, y = 2mn, z = m^2 + n^2$. If one of m or n is even, then y is divisible by 4. Suppose n, m are both odd. We may write $n = 2s + 1$ and $m = 2t + 1$. Thus,

$$x = m^2 - n^2 = (2s + 1)^2 - (2t + 1)^2 = 4s^2 + 4s - 4t^2 - 4t,$$

showing that x is divisible by 4.